Successive Improvement of the Modified Differential Approximation in Radiative Heat Transfer

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A modified differential approximation proposed by Olfe is extended to the treatment of a medium with isotropic scattering and boundary reflection. The method can be used to readily generate successive improvement of the solutions. In this work, the basic procedure is demonstrated by computing the transmissivity and hemispherical reflectivity of an absorbing, isotropically scattering slab with a reflecting boundary. Comparison of present computational results with existing exact solutions shows that, in general, the modified differential approximation is superior to the unmodified differential approximation, and that one or two applications of the improvement of the modified differential approximation produces results very close to the exact solution. The present method also gives accurate solutions for an optically-thin medium. The general concept of the current method appears to be adaptable to the treatment of multidimensional cases.

Nomenclature

E_n = exponential integral of order n
G = zeroth moment of I, Eq. (10)
I = radiative intensity
i = subscript that denotes the contribution of incident radius
I_o = diffuse intensity incident on the boundary
K = backward half-range flux at τ_c, Eq. (8)
ka = superscript that denotes the kth part of the approximate solution
ke = superscript that denotes the kth part of the exact solution
m = subscript that denotes the contribution of the medium
R = hemispherical reflectivity, Eq. (3)
S = source function
T = transmissivity, Eq. (4)
μ = cosine of the angle between the direction of radiative intensity and the positive τ_z axis
ρ_d = diffuse reflectivity
τ_c = optical thickness of the slab
τ_p = arbitrary optical path from the surface to a location in the medium
τ_z = optical variable in the direction perpendicular to the surface of medium
Ω = solid angle or direction in space
ω = albedo

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Introduction

THE formulation of radiative transfer in an absorbing, emitting, and scattering medium produces an integro-differential equation which is often subject to boundary conditions with integral terms. Because of the mathematical complexity of the formulation, a large variety of approximation methods have been proposed. General reviews of those methods have been given by many authors, for example, Özisik[1] and Siegel and Howell,[2] etc. Because of its simple and general nature, the differential approximation is one of the most popular approximation methods. To increase accuracy, many modifications and higher-order revisions have been proposed.[3-10] Although a method which is simple, accurate, and applicable for general cases does not exist, a modified differential approximation (MDA) proposed by Olfe[3] appears to meet most of these requirements when it is applied to the radiative transfer in a medium with absorbing and emitting, but without scattering and boundary reflection.[3-6] In the present work, the MDA is generalized to solve the radiative transfer in a medium with scattering and boundary reflection. Moreover, it is found that the integro-differential equation which is solved approximately in the MDA can be divided into two problems: one has an exact solution, and the other is mathematically equivalent to the original integro-differential equation, i.e., this new integro-differential equation can be divided further or be solved by the MDA. Since the approximate solution of this new integro-differential equation has less contribution to the total solution than the original one, this technique can provide a successive improvement to the solution of the MDA.

In the present work, the hemispherical reflectivity and transmissivity of an absorbing, isotropically scattering slab with diffusely reflecting boundaries are computed by the aforementioned methods. Because of its engineering importance, a variety of methods,[11-15] including the exact solution given by Lii and Özisik,[11] have been used to solve the present problem. This problem is chosen because these existing solutions can be used to examine the effectiveness of the present method.

Analysis

The intensity along a path in an absorbing, emitting, and isotropically scattering medium with homogeneous properties can be expressed as

$$I(\tau_p,\Omega) = I_0(\tau_{p0},\Omega)e^{-(\tau_p-\tau_{p0})} + \int_{\tau_{p0}}^{\tau_p} S(\tau_p')e^{-(\tau_p-\tau_p')}\mathrm{d}\tau_p' \quad (1)$$

where τ_p denotes an arbitrary optical path from the surface, Ω the direction of propagation of radiation, I_0 incident radiation from location τ_{p0} on the boundary of the medium, and S the source function. Since the first term of Eq. (1) is expressed exactly and explicitly for known boundaries, Olfe[3] suggested that to approximate only the second term may increase the accuracy of solution. Note that the whole intensity is approximated in the unmodified differential approximation. However, for a medium with reflecting boundaries, the intensity is not given explicitly at the boundaries. Thus, to take the advantage of the MDA, we need to treat the intensity at the boundaries first. It is found that a superposition method proposed by Özisik and Sutton[14] may be used to simplify this kind of boundary condition. Note that this superpositioning of boundaries is not generally applicable to multidimensional problems; however, all other features of the current procedure will work for at least specified boundary intensities in such problems.

In this work, Olfe's MDA will be generalized to obtain the hemispherical reflectivity and transmissivity of an absorbing, isotropically scattering, nonconservative, plane-parallel slab. The optical thickness of the slab is τ_c. The boundary surface at $\tau_z=0$ is transparent and is exposed to unit diffuse intensity, and the boundary at $\tau_z=\tau_c$ is partially reflecting having diffuse reflectivity, ρ_d. Also, it is assumed that the emission from the medium and from the second boundary is negligible. The resultant problem is described by

$$\mu\frac{\partial I(\tau_z,\mu)}{\partial\tau_z} + I(\tau_z,\mu) = \frac{\omega}{2}\int_{-1}^{1} I(\tau_z,\mu')\mathrm{d}\mu' \quad 0\le\tau_z\le\tau_c,$$

$$|\mu|\le 1 \quad (2a)$$

$$I(0,\mu)=1,\ \mu>0 \quad (2b)$$

$$I(\tau_c,-\mu) = 2\rho_d\int_0^1 I(\tau_c,\mu')\mu'\mathrm{d}\mu',\ \mu>0 \quad (2c)$$

where τ_z denotes the optical variable in the direction perpendicular to the surface of the medium, μ the cosine of the angle between the direction of radiative intensity and the positive τ_z axis, and ω the single scattering albedo. The hemispherical reflectivity and the transmissivity of the slab are defined, respectively, as

$$R = 2\int_0^1 I(0,-\mu)\mu\mathrm{d}\mu \quad (3)$$

$$T = 2\int_{-1}^1 I(\tau_c,\mu)\mu\mathrm{d}\mu \quad (4)$$

Following the previous discussion, the complete problem [Eq. (2)] is split into two sub-problems, in which the intensity is unity or zero on boundary before we apply the MDA. Then the problem becomes

$$\mu\frac{\partial I_1(\tau_z,\mu)}{\partial\tau_z} + I(\tau_z,\mu) = S_1(\tau_z), 0\le\tau_z\le\tau_c,\ |\mu|\le 1 \quad (5a)$$

$$I_1(0,\mu)=1,\mu>0 \quad (5b)$$

$$I_1(\tau_c,-\mu)=0,\mu>0 \quad (5c)$$

$$S_1(\tau_z) = \frac{\omega}{2}\int_{-1}^1 I_l(\tau_z,\mu')\mathrm{d}\mu' \quad (5d)$$

and

$$\mu\frac{\partial I_2(\tau_z,\mu)}{\partial\tau_z} + I_2(\tau_z,\mu) = S_2(\tau_z), 0\le\tau_z\le\tau_c,\ |\mu|\le 1 \quad (6a)$$

$$I_2(0,\mu)=0,\mu>0 \quad (6b)$$

$$I_2(\tau_c,-\mu)=1,\mu>0 \quad (6c)$$

$$S_2(\tau_z) = \frac{\omega}{2}\int_{-1}^1 I_2(\tau_z,\mu')\mathrm{d}\mu' \quad (6d)$$

The total radiative intensity in the original problem becomes

$$I(\tau_z,\mu) = I_1(\tau_z,\mu) + 2\rho_d K I_2(\tau_z,\mu) \quad (7)$$

where

$$K = \left[\int_0^1 I_1(\tau_c,\mu)\mu\mathrm{d}\mu\right] \Big/ \left[1-2\rho_d\int_0^1 I_2(\tau_c,\mu)\mu\mathrm{d}\mu\right] \quad (8)$$

Note that now the intensity at the boundaries is given explicitly in both subproblems, so the MDA is applicable for each problem.

These problems are identical, therefore we will only consider the solution of I_1. Now, I_1 can be written in terms of two components, the direct boundary and medium contributions as

$$I_1(\tau_z,\mu) = I_{1i}(\tau_z,\mu) + I_{1m}(\tau_z,\mu),\ |\mu|\le 1 \quad (9a)$$

where

$$I_{1i}(\tau_z,\mu) = e^{-(\tau_z/\mu)},\mu>0, \qquad I_{02(\tau_z},\mu) = 0,\mu<0 \quad (9b)$$

$$I_{1m}(\tau_z,\mu) = \int_0^{\tau_z}\frac{1}{\mu}S_1(\tau_z',\mu)e^{-(\tau_z-\tau_z')/\mu}\mathrm{d}\tau_z',\mu>0$$

$$I_{1m}(\tau_z,\mu) = -\int_{\tau_z}^{\tau_c}\frac{1}{\mu}S_1(\tau_z',\mu)e^{-(\tau_z-\tau_z')/\mu}\mathrm{d}\tau_z',\mu<0 \quad (9c)$$

The I_1 problem can then be divided into a differential equation for I_{1i}, whose solution is expressed as Eq. (9b), and an integro-differential equation for I_{1m}. The latter is

$$\mu\frac{\partial I_{1m}(\tau_z,\mu)}{\partial\tau_z} + I_{1m}(\tau_z,\mu) = \frac{\omega}{4\pi}[G_{1m}(\tau_z) + G_{1i}(\tau_z)]$$

$$0\le\tau_z\le\tau_c,\ |\mu|\le 1$$

$$I_{1m}(0,\mu)=0,\mu>0$$

$$I_{1m}(\tau_c,-\mu)=0,\mu>0 \quad (10)$$

where

$$G_{1m}(\tau_z) = 2\pi\int_{-1}^1 I_{1m}(\tau_z,\mu)\mathrm{d}\mu$$

$$G_{1i}(\tau_z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\tau_p}}{\tau_p^3}\tau_z\mathrm{d}\tau_x'\mathrm{d}\tau_y'$$

$$\tau_p = \sqrt{(0-\tau_x')^2 + (0-\tau_y')^2 + (\tau_z-0)^2}$$

Note that here we use the three-dimensional optical variable so that minimal modification is required to move to multidimensional problems. Following Özisik,[1] we obtain the differential approximation of Eq. (10) as

$$\frac{\mathrm{d}^2 G_{1m}(\tau_z)}{\mathrm{d}\tau_z^2} - 3(1-\omega)G_{1m}(\tau_z) = -3\omega G_{1i}(\tau_z) \quad 0 \le \tau_z \le \tau_c \tag{11}$$

where

$$G_{1m}(0) = \frac{2}{3}\frac{\mathrm{d}G_{1m}(0)}{\mathrm{d}\tau_z} \qquad G_{1m}(\tau_c) = -\frac{2}{3}\frac{\mathrm{d}G_{1m}(\tau_c)}{\mathrm{d}\tau_z}$$

and for the slab

$$G_{1i} = 2\pi E_2(\tau_z)$$

A closed-form solution of Eq. (11) can be obtained by the method of Green's functions

$$G_{1m}(\tau_z) = \int_0^{\tau_z} g_1(\tau_z,\xi)[-3\omega G_{1i}(\xi)]\mathrm{d}\xi + \int_{\tau_z}^{\tau_c} g_2(\tau_z,\xi) \times [-3\omega G_{1i}(\xi)]\mathrm{d}\xi \tag{12a}$$

where

$$g_1(\tau_z,\xi) = \left[\left(e^{\lambda\xi} - \frac{1-\gamma\lambda}{1+\gamma\lambda}e^{-\lambda\xi}\right)\left(e^{\lambda\tau_z} - \frac{1-\alpha\lambda}{1+\alpha\lambda}e^{2\lambda\tau_c-\lambda\tau_z}\right)\right] \Big/ \lambda D(\tau_z), \qquad 0 \le \xi \le \tau_z \tag{12b}$$

$$g_2(\tau_z,\xi) = \left[\left(e^{\lambda\tau_z} - \frac{1-\gamma\lambda}{1+\gamma\lambda}e^{-\lambda\tau_z}\right)\left(e^{\lambda\xi} - \frac{1-\alpha\lambda}{1+\alpha\lambda}e^{2\lambda\tau_c-\lambda\xi}\right)\right] \Big/ \lambda D(\tau_z), \qquad \tau_z \le \xi \le \tau_c \tag{12c}$$

and

$$D(\tau_z) = \left(e^{\lambda\tau_z} - \frac{1-\gamma\lambda}{1+\gamma\lambda}e^{-\lambda\tau_z}\right)\left(e^{\lambda\tau_z} + \frac{1-\alpha\lambda}{1+\alpha\lambda}e^{2\lambda\tau_c-\lambda\tau_z}\right) - \left(e^{\lambda\tau_z} + \frac{1-\gamma\lambda}{1+\gamma\lambda}e^{-\lambda\tau_z}\right)\left(e^{\lambda\tau_z} - \frac{1-\alpha\lambda}{1+\alpha\lambda}e^{2\lambda\tau_c-\lambda\tau_z}\right) \tag{12d}$$

where $\alpha = 2/3$, $\gamma = -2/3$, and $\lambda = 3\sqrt{(1-\omega)}$. Similarly, one can solve the I_2 problem. Then the reflectivity and the transmissivity are determined from the relations

$$R = 2\left[\frac{G_{1m}(0)}{4\pi} + 2\rho_d K \frac{G_{2m}(0)}{4\pi} + 2\rho_d K E_3(\tau_c)\right] \tag{13}$$

$$T = 2K(1-\rho_d) \tag{14}$$

$$K = \left[\frac{G_{1m}(\tau_c)}{4\pi} + E_3(\tau_c)\right] \Big/ \left(1 - 2\rho_d \frac{G_{2m}}{4\pi}\right) \tag{15}$$

Reconsidering the I_{1m} Eq. (10), one finds that the radiative intensity incident on both boundaries is zero. This characteristic implies that a successive improvement of the solution is possible. Again, I_{1m} is divided into an exact I_{1m}^{1e} and an approximate I_{1m}^{1a} part, as

$$I_{1m} = I_{1m}^{1e} + I_{1m}^{1a} \tag{16}$$

where the superscripts $1a$ and $1e$ denote the first part of the approximate solution and the first part of the exact solution, respectively. Then Eq. (10) becomes

$$\mu \frac{\partial I_{1m}^{1e}}{\partial \tau_z} + I_{1m}^{1e} = \frac{\omega}{4\pi} G_{1i}, 0 \le \tau_z \le \tau_c, |\mu| \le 1$$

$$I_{1m}^{1e}(0,\mu) = 0, \mu > 0, \qquad I_{1m}^{1e}(\tau_c,\mu) = 0, \mu > 0 \tag{17}$$

and

$$\mu \frac{\partial I_{1m}^{1a}}{\partial \tau_z} + I_{1m}^{1a} = \frac{\omega}{4\pi}[G^{1m} + G_{1m}^{1e}], 0 \le \tau_z \le \tau_c, \ |\mu| \le 1$$

$$I_{1m}^{1a}(0,\mu) = 0, \mu > 0, \qquad I_{1m}^{1a}(\tau_c - \mu) = 0, \mu > 0 \tag{18}$$

where

$$G_{1m}^{1a}(\tau_z) = 2\pi \int_{-1}^{1} I_{1m}^{1a}(\tau_z,\mu)\mathrm{d}\mu$$

$$G_{1m}^{1e}(\tau_z) = 2\pi \int_{-1}^{1} I_{1m}^{1e}(\tau_z,\mu)\mathrm{d}\mu$$

which is obtained from the solution of Eq. (17). It is worth noting that Eq. (17) has an exact solution, and Eq. (18) has the same mathematical form as Eq. (10). Therefore, Eq. (18) can

Table 1 Slab reflectivity, transparent boundaries, $I_0 = 1.0$ at $\tau_z = 0$

ω	Method	$\tau_c = 0.1$	$\tau_c = 1.0$	$\tau_c = 5.0$
0.9	P-1 approximation	0.0559	0.3337	0.4635
	MDA	0.0748	0.3567	0.4862
	First improvement of MDA	0.0745	0.3550	0.4827
	Second improvement of MDA	0.0744	0.3538	0.4809
	Third improvement of MDA	0.0744	0.3528	0.4793
	Exact solution	0.0744	0.3527	0.4763
0.5	P-1 approximation	0.0221	0.0924	0.1010
	MDA	0.0400	0.1440	0.1561
	First improvement of MDA	0.0386	0.1373	0.1498
	Second improvement of MDA	0.0384	0.1350	0.1476
	Third improvement of MDA	0.0384	0.1343	0.1468
	Exact solution	0.0384	0.1342	0.1465
0.1	P-1 approximation	a	a	a
	MDA	0.0077	0.0230	0.0237
	First improvement of MDA	0.0072	0.0209	0.0219
	Second improvement of MDA	0.0072	0.0208	0.0217
	Third improvement of MDA	0.0072	0.0207	0.0217
	Exact solution	0.0072	0.0207	0.0217

[a]Negative value.

Table 2 Slab reflectivity: reflecting boundary at $\tau_z = \tau_c$, transparent boundary at $\tau_z = 0$, $I_0 = 1.0$ at $\tau_z = 0$

ω	Method	$\tau_c = 0.1$	$\tau_c = 1.0$	$\tau_c = 5.0$
0.9	P-1 approximation	0.9608	0.6919	0.4683
	MDA	0.9636	0.7187	0.4914
	First improvement of MDA	0.9615	0.7117	0.4884
	Second improvement of MDA	0.9612	0.7052	0.4865
	Third improvement of MDA	0.9612	0.7014	0.4850
	Exact solution	0.9612	0.7009	0.4818
0.5	P-1 approximation	0.8190	0.1857	0.1010
	MDA	0.8320	0.2557	0.1561
	First improvement of MDA	0.8269	0.2486	0.1499
	Second improvement of MDA	0.8264	0.2444	0.1476
	Third improvement of MDA	0.8264	0.2430	0.1468
	Exact solution	0.8264	0.2428	0.1465
0.1	P-1 approximation	0.6972	a	a
	MDA	0.7190	0.0779	0.0237
	First improvement of MDA	0.7173	0.0759	0.0219
	Second improvement of MDA	0.7173	0.0757	0.0217
	Third improvement of MDA	0.7173	0.0757	0.0217
	Exact solution	0.7168	0.0756	0.0217

[a]Negative value.

be solved by the differential approximation or be solved by further dividing it into two equations.

The general expressions of hemispherical reflectivity and transmissivity for the kth improvement are as follows:

$$R=2\left\{\frac{\omega}{4\pi}\int_0^{\tau_c}\left[G_{1i}(\tau_z')+2\rho_d K G_{2i}(\tau_z')\right]E_2(\tau_z'-0)\mathrm{d}\tau_z' + \frac{G_{1m}^{1a}(\tau_c)}{4\pi}+\frac{G_{1a}^{1m}(0)}{4\pi}+2\rho_d K E_3(\tau_c)\right\} \tag{19a}$$

$$T=2(1-\rho_d)K \tag{19b}$$

in which K satisfies

$$K=\frac{\omega}{4\pi}\int_0^{\tau_c}\left[G_{1i}(\tau_z')+2\rho_d K G_{2i}\right]E_2(\tau_z-\tau_z')\mathrm{d}\tau_z' + \frac{G_{1m}^{1a}(0)}{4\pi}+\frac{G_{1m}^{1a}(\tau_c)}{4\pi}+E_3(\tau_c), \quad k=1$$

and

$$R=\frac{\omega}{4\pi}2\int_0^{\tau_c}\left\{G_{1i}(\tau_z')+2\rho_d K G_{2i}(\tau_z') + \sum_{j=2}^{k}\left[G_{1m}^{(j-1)e}(\tau_z')+2\rho_d K G_{2m}^{(j-1)e}(\tau_z')\right]\right\}E_2(\tau_z'-0)\mathrm{d}\tau_z' + 2\left\{\frac{\omega}{4\pi}\left[G_{1m}^{ka}(0)+2\rho_d K G_{2m}^{ka}(0)\right]+2\rho_d K E_3(\tau_c)\right\} \tag{20a}$$

$$T=2(1-\rho_d)K \tag{20b}$$

in which K satisfies

$$K=\frac{\omega}{4\pi}\int_0^{\tau_c}\left\{G_{1i}(\tau_z')+2\rho_d K G_{2i}(\tau_z') + \sum_{j=2}^{k}\left[G_{1m}^{(j-1)e}(\tau_z')+2\rho_d K G_{2m}^{(j-1)e}(\tau_z')\right]\right\}E_2(\tau_c-\tau_z')\mathrm{d}\tau_z' + \frac{\omega}{4\pi}\left[G_{1m}^{ka}(\tau_c)+2\rho_d K G_{2m}^{ka}(\tau_c)\right]+2\rho_d K E_3(\tau_c), \quad k\geq 2$$

From the above expressions, one may find that the exact part increases as k increases. Hence, this procedure will lead the approximate solution to the exact solution as $k\to\infty$.

Results and Discussion

Tables 1-3 show the computational results of the hemispherical reflectivity and the transmissivity of the slab for several different values of optical thickness, single scattering albedo and boundary surface reflectivity. The results listed are obtained by the MDA and the first, second, and third improvement of the MDA. The exact results and P-1 approximation results from Lii and Özisik[11] are also listed in Tables 1-3.

In general, the unmodified differential approximation (equivalent to P-1) is inaccurate near the boundary of a medium exposed to external radiation. Here, the definition of the hemispherical reflectivity and transmissivity of the slab contains the radiative flux on the boundary. Thus, the application of the present methods to this case is an important extension. Lii and Özisik[11] have pointed out that the P-1 approximation gives reasonably good results for large optical thickness and albedo close to one, but the accuracy of the approximation is not so good for smaller values of albedo. From Tables 1-3, one can see that the accuracy of the results is improved by using the MDA, especially when the albedo is small.

Table 3 Slab transmissivity, transparent boundaries, $I_0=1.0$ at $\tau_z=0$

ω	Method	$\tau_c=0.1$	$\tau_c=1.0$	$\tau_c=5.0$
0.9	P-1 approximation	0.9204	0.4885	0.0507
	MDA	0.9069	0.4887	0.0547
	First improvement of MDA	0.9060	0.4796	0.0543
	Second improvement of MDA	0.9060	0.4765	0.0544
	Third improvement of MDA	0.9059	0.4750	0.0544
	Exact solution	0.9060	0.4747	0.0544
0.5	P-1 approximation	0.8828	0.2911	0.0022
	MDA	0.8722	0.3182	0.0046
	First improvement of MDA	0.8706	0.3099	0.0050
	Second improvement of MDA	0.8704	0.3075	0.0052
	Third improvement of MDA	0.8704	0.3068	0.0052
	Exact solution	0.8704	0.3067	0.0053
0.1	P-1 approximation	0.8480	0.1930	0.0001
	MDA	0.8402	0.2339	0.0020
	First improvement of MDA	0.8397	0.2320	0.0020
	Second improvement of MDA	0.8396	0.2319	0.0020
	Third improvement of MDA	0.8396	0.2318	0.0020
	Exact solution	0.8394	0.2317	0.0020

In this work, the MDA has been extended to the treatment of a medium with boundary reflection. The results of an example ($\rho_d=1.0$) are listed in Table 2. It has been found that the accuracy of the ordinary differential approximation may be better than that of the MDA for a few situations (e.g., $\omega=0.9$, $\rho_d=1.0$, and $\tau_c=0.1$). However, in general, the MDA is superior to the ordinary differential approximation.

The comparison of the improvement of the generalized MDA with the exact solution and other methods can be made in Tables 1-3. It is found that the results obtained by using the improvement one or two times are very close to those obtained by the exact solution and are always far better than those obtained by the unmodified P-1 approximation. This comparison demonstrates the effectiveness of the current improvement of the MDA.

In addition to the exact solution and the P-1 approximation,[11] many authors have studied the present example using many different approximate methods.[12-15] These approximate methods provide results for optical thickness 0.5 or larger. That is, the results for small optical thickness (≤ 0.1) are not available. However, Tables 1-3 show that the current improvement of the generalized MDA works quite well for these cases as well. This is an important advantage of the present method over the other approximation methods.

Conclusions

The conclusions of the present paper can be summarized as follows:

1) Olfe's MDA is extended to the treatment of a medium with isotropic scattering and boundary reflection.

2) An improvement of the MDA is proposed in the present work. It can be used to readily generate a successive improvement of solutions.

3) The hemispherical reflectivity and transmissivity of an absorbing, isotropically scattering slab with a reflecting boundary are computed by the MDA and its improvement. The results of this computation demonstrate the effectiveness of the present methods.

4) The results of the current example show that, in general, the MDA is superior to the unmodified differential approximation, one or two applications of the improvement of the MDA may produce results very close to the exact solution. Additionally the successive improvement approach has the advantage that it generates accurate results for an optically thin medium.

5) Since similar formulations can be derived for multidimensional cases, the current methods appear to be adaptable to the treatment of those cases.

References

[1]Özisik, M. N., *Radiative Transfer*, John Wiley and Sons, New York, NY, 1973.

[2]Siegel, R. and Howell, J. R., *Thermal Radiation Heat Transfer*, 2nd ed., McGraw-Hill, New York, NY, 1980.

[3]Olfe, D. B., "A Modification of Differential Approximation for Radiative Transfer," *AIAA Journal*, Vol. 5, 1967, pp. 638-643.

[4]Olfe, D. B., "Application of the Modified Differential Approximation to Radiative Transfer in a Gray Medium between Concentric Cylinders and Spheres," *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 8, 1968, pp. 899-907.

[5]Olfe, D. B., "Radiative Equilibrium of a Gray Medium Bounded by Nonisothermal Walls," *Progress in Astronautics and Aeronautics*, Vol. 23, 1970, pp. 295-317.

[6]Glatt, L. and Olfe, D. B., "Radiative Equilibrium of a Gray Medium in a Rectangular Enclosure," *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 13, 1973, pp. 881-895.

[7]Modest, M. F., "Two-Dimensional Radiative Equilibrium of a Gray Medium in a Plane Layer Bounded by Gray Nonisothermal Walls," ASME *Journal of Heat Transfer*, Vol. 96, 1974, pp. 483-488.

[8]Yuen, W. W. and Tien, C.L., "A Successive Approximation Approach to Problems in Radiative Transfer with a Differential Formulation," ASME *Journal of Heat Transfer*, Vol. 102, 1980, pp. 86-91.

[9]Bayzitoglu, Y. and Higenyi, J., "The Higher-Order Differential Equations of Radiative Transfer: P Approximation," *AIAA Journal*, Vol. 17, 1979, pp. 424-431.

[10]Ratzel, A. C. III, "Two-Dimensional Radiation in Absorbing-Emitting Medium Using the P-N Approximation," ASME *Journal of Heat Transfer, Vol. 105, 1983, pp. 333-340.*

[1]Lii, C. C. and Özisik, M. N., "Hemispherical Reflectivity and Transmissivity of a Absorbing, Isotropically Scattering Slab with a Reflecting Boundary," *International Journal of Heat and Mass Transfer*, Vol. 16, 1973, pp. 685-690.

[12]Sutton, W. H. and Özisik, M. N., "An Iterative Solution for Anisotropic Radiative Transfer in a Slab," ASME *Journal of Heat Transfer*, 1979, Vol. 101, pp. 695-698.

[13]Sutton, W. H. and Özisik, M. N., "A Fourier Transform Solution for Radiative Transfer in a Slab with Isotropic Scattering and Boundary Reflection," *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 22, 1979, pp. 55-64.

[14]Özisik, M. N. and Sutton, W. H., "A Source Function Expansion in Radiative Transfer," ASME *Journal of Heat Transfer*, Vol. 102, 1980, pp. 715-718.

[15]Özisik, M. N. and Yener, Y., "The Galerkin Method for Solving Radiation Transfer in Plane-Parallel Participating Media," ASME *Journal of Heat Transfer*, Vol. 104, 1982, pp. 351-354.

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